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A VARIATIONAL APPROACH TO THE PROBLEM OF STATIONARY LAMINAR CONVECTION WITH DISSIPATION IN THE VARIABLE PROPERTY FLUID

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Abstract—A variational principle of Hamilton's has been used to solve the problem of stationary convection with dissipation in the variable property fluid flowing over a flat plate with unheated starting length. Thermal diffusivity and viscosity were assumed to be linear functions of temperature. Two ordinary differential equations were derived from the variational formulation and then numerically solved for Eckert numbers ranging from 0 to 2 using a digital computer. The results show that the value of the Eckert number has a considerable influence on heat transfer, while the change of friction factor is negligible. For higher values of Eckert number, at a certain distance from the edge of the plate, the thermal boundary layer becomes thicker than the momentum boundary layer even for a fluid with a Prandtl number greater than unity.

NOMENCLATURE

A, constant defined by equation (28);
a, thermal diffusivity;
B, constant defined by equation (28);
c_f, skin friction coefficient;
C_p, specific heat;
Ec, Eckert number;
f, momentum boundary-layer thickness;
l, characteristic length of the plate;
Nu, local Nusselt number;
Pr, Prandtl number;
Re, Reynolds number;
T, temperature;
U_∞, free stream velocity;
u, velocity in the x-direction;
v, velocity in the y-direction;
x_0, unheated starting length;
x, y, Cartesian coordinates;
Y, thermal to momentum boundary-layer thickness ratio (Δ/f).

Greek symbols

Δ, thermal boundary-layer thickness;
δ, variational notation;
η, λ, dimensionless y-coordinates ($\eta = y/\Delta$, $\lambda = y/f$);
μ, dynamic viscosity;
μ_1, Lagrange's multiplier;
ν, kinematic viscosity;
ξ, dimensionless x-coordinate ($\xi = x/l$);
ρ, density;
φ, dimensionless variable ($\varphi = f^2/l^2$).

Subscripts

x, y, partial or ordinary derivatives with respect to x, y;
w, wall property;
∞, free stream property.

Superscripts

$*$, dimensionless variable.

INTRODUCTION

THE OBJECT of this paper is to present a variational approach to the problem of stationary convection in the variable property fluid using a variational formulation of Hamilton's type introduced by Vujanović [1–3]. According to this formulation the Lagrange's density, containing an additional physically meaningless parameter, is introduced into the action integral. When the first variation of the action integral is set equal to zero, the complex differential equations, containing the parameter mentioned above, are obtained. However, in the case of the limiting transition, by which this parameter approaches to zero, the exact differential equations of the considered process are obtained. It has to be noted here that this variational formulation is especially convenient for the direct variational method — the Kantorovich's method of partial integration.

The theory of chain-systems* given by Arzhanykh [4] will be used. Although the theory of Arzhanykh concerns only the discrete dynamics, in this paper it will be extended and applied to the processes being described by partial differential equations.

VARIATIONAL PRINCIPLE

We shall consider the stationary laminar flow of incompressible fluid with variable viscosity ν and thermal diffusivity a, over a semi-infinite flat plate, taking into account the mechanical energy dissipation caused by viscosity. The values ν and a are considered to be linear functions of temperature. The temperature of the plate T_w is considered constant.

Using these assumptions, the two-dimensional flow in the laminar boundary layer without pressure gradient is described by the following momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right). \tag{1}$$

The boundary-layer energy equation is of the form

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(a\frac{\partial T}{\partial y}\right) + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{2}$$

Equations (1) and (2) along with the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

and appropriate boundary conditions, when the functions $a(T)$ and $\nu(T)$ are known, make possible evaluation of three unknown functions $u(x,y)$; $v(x,y)$ and $T(x,y)$. The boundary conditions for the problem under consideration are:

$$\begin{aligned} y = 0&: \quad u = v = 0; \quad T = T_w \quad \text{and} \\ y = y_m&: \quad u = U_\infty \qquad T = T_\infty, \end{aligned} \tag{4}$$

where $y_m = f(x)$ on $y_m = \Delta(x)$, $f(x)$ and $\Delta(x)$ being momentum and thermal boundary-layer thicknesses respectively.

In order to apply the theory of chain-systems two Lagrangians are made:

$$L^{(1)} = \left\{m\left[\frac{1}{2}u\left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right] - \frac{\nu}{2}\left(\frac{\partial u}{\partial y}\right)^2\right\}e^{x/m} + \mu_1\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{5}$$

$$L^{(2)} = \left\{m\left[\frac{1}{2}u\left(\frac{\partial T}{\partial x}\right)^2 + v\frac{\partial T}{\partial x}\frac{\partial T}{\partial y} - \frac{1}{2}\frac{\partial a}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2\frac{\partial T}{\partial x} - \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial T}{\partial x}\right] - \frac{a}{2}\left(\frac{\partial T}{\partial y}\right)^2\right\}e^{x/m}. \tag{6}$$

Lagrange's multiplier $\mu_1 = \mu_1(x,y)$ in the first Lagrangian is unknown. The constant parameter m is physically meaningless.

According to the theory of chain systems functions u, v, T and μ_1 are generalized coordinates and they are divided into two groups. Functions u, v and μ_1 belong to the first group and the corresponding Lagrangian is $L^{(1)}$, while temperature T is in the second group with Lagrangian $L^{(2)}$. The corresponding action integrals are:

$$I_1 = \int_{x_0}^{x=l}\int_0^{y_m} L^{(1)}\,dx\,dy \tag{7}$$

$$I_2 = \int_{x_0}^{x=l}\int_0^{y_m} L^{(2)}\,dx\,dy. \tag{8}$$

Differential equations (1)–(3), describing thermal and momentum boundary layer over the flat plate, can be derived from the variational principle:

$$\delta I_1 = 0 \tag{9}$$

$$\delta I_2 = 0 \tag{10}$$

It has to be noted here that the variation of the first action integral (7) is related only to the first group of coordinates (u, v and μ_1), while the variation of the second action integral (8) is related only to the second group of coordinates (T).

After varying the action integrals (7) and (8) and partial integrating we obtain:

$$\begin{aligned} &\int_{x_0}^{x=l}\left|\frac{\partial I^{(1)}}{\partial u_y}\delta u\right|_0^{y_m}dx + \int_0^{y_m}\left|\frac{\partial L^{(1)}}{\partial u_x}\delta u\right|_{x_0}^{x=l}dy \\ &+ \int_{x_0}^{x=l}\left|\frac{\partial L^{(1)}}{\partial v_y}\delta v\right|_0^{y_m}dx \\ &+ \int_{x_0}^{x=l}\int_0^{y_m}\left[\frac{\partial L^{(1)}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial L^{(1)}}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial L^{(1)}}{\partial u_y}\right]\delta u\,dx\,dy \\ &+ \int_{x_0}^{x=l}\int_0^{y_m}\left[\frac{\partial L^{(1)}}{\partial v} - \frac{\partial}{\partial x}\frac{\partial L^{(1)}}{\partial v_x} - \frac{\partial}{\partial y}\frac{\partial L^{(1)}}{\partial v_y}\right]\delta v\,dx\,dy \\ &+ \int_{x_0}^{x=l}\int_0^{y_m}\frac{\partial L^{(1)}}{\partial \mu_1}\delta\mu_1\,dx\,dy = 0 \end{aligned} \tag{11}$$

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* According to this theory, the chain-system was titled the mechanical system, the generalized coordinates of which can be divided into k groups:

$$q_{i1}, q_{i2}, \ldots, q_{in_i} \qquad (i = 1, 2, \ldots, k)$$

with k partial Lagrangians $L^{(i)}$ ($i = 1, \ldots, k$) which, in general, depend on time, all generalized coordinates and all generalized velocities, so that equations:

$$\frac{d}{dt}\frac{\partial L^{(i)}}{\partial \dot{q}_{i\alpha}} - \frac{\partial L^{(i)}}{\partial q_{i\alpha}} = 0 \qquad \begin{matrix}(i = 1, 2, \ldots, k)\\(\alpha = 1, 2, \ldots, n_i)\end{matrix}$$

give the differential equations of considered system.

$$\int_{x_0}^{x=l}\left|\frac{\partial L^{(2)}}{\partial T_y}\delta T\right|_0^{y_m}\mathrm{d}x+\int_0^{y_m}\left|\frac{\partial L^{(2)}}{\partial T_x}\delta T\right|_{x_0}^{x=l}\mathrm{d}y$$
$$+\int_{x_0}^{x=l}$$
$$-\frac{\partial}{\partial y}\frac{\partial L^{(2)}}{\partial T_y}\bigg]\delta T\mathrm{d}x\mathrm{d}y=0. \quad (12)$$

Since velocity u and temperature T are determined at all boundaries except at $x = l$, first members in equations (11) and (12) are equal to zero (because $\delta u = 0$ and $\delta T = 0$).

If at $x = l$ and $y = y_m$ for arbitrary values of variations δu, δv and δT, the following conditions

$$\frac{\partial L^{(1)}}{\partial u_x}\delta u\bigg|_{x=l}=0;\quad \frac{\partial L^{(1)}}{\partial v_y}\delta v\bigg|_{y=y_m}=0\quad\text{and}$$
$$\frac{\partial L^{(2)}}{\partial T_x}\delta T\bigg|_{x=l}=0 \quad (13)$$

are satisfied, the variational equations (11) and (12) result in Euler–Lagrange's equations:

$$\frac{\partial L^{(1)}}{\partial u}-\frac{\partial}{\partial x}\frac{\partial L^{(1)}}{\partial u_x}-\frac{\partial}{\partial y}\frac{\partial L^{(1)}}{\partial u_y}=0$$
$$\frac{\partial L^{(1)}}{\partial v}-\frac{\partial}{\partial x}\frac{\partial L^{(1)}}{\partial v_x}-\frac{\partial}{\partial y}\frac{\partial L^{(1)}}{\partial v_y}=0 \quad (14)$$
$$\frac{\partial L^{(1)}}{\partial \mu_1}=0$$
$$\frac{\partial L^{(2)}}{\partial T}-\frac{\partial}{\partial x}\frac{\partial L^{(2)}}{\partial T_x}-\frac{\partial}{\partial y}\frac{\partial L^{(2)}}{\partial T_y}=0.$$

Substituting Lagrangians (5) and (6) into corresponding equations (13) and (14) and dividing by $e^{x/m}$, we get

$$m\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\mu_1 e^{-x/m}\right)\delta u\bigg|_{x=l}=0$$
$$\mu_1\delta v\bigg|_{y=y_m}=0 \quad (15)$$
$$m\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}-\frac{1}{2}\frac{\partial a}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2-\frac{v}{C_p}\left(\frac{\partial u}{\partial y}\right)^2\right]\delta T\bigg|_{x=l}=0$$

$$m\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-m\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial y}\right)$$
$$-m\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)-\mu_1 e^{-x/m}=0 \quad (16)$$

$$m\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-\mu_1 e^{-x/m}=0$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (17)$$

$$m\frac{\partial}{\partial T}\left[\frac{1}{2}u\left(\frac{\partial T}{\partial x}\right)^2+v\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}-\frac{1}{2}\frac{\partial a}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2\frac{\partial T}{\partial x}-\frac{v}{C_p}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial T}{\partial x}\right]-\frac{1}{2}\frac{\partial a}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2$$
$$-u\frac{\partial T}{\partial x}-v\frac{\partial T}{\partial y}+\frac{1}{2}\frac{\partial a}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2$$
$$+\frac{v}{C_p}\left(\frac{\partial u}{\partial y}\right)^2-m\frac{\partial}{\partial x}\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}-\frac{1}{2}\frac{\partial a}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2-\frac{v}{C_p}\left(\frac{\partial u}{\partial y}\right)^2\right]-m\frac{\partial}{\partial y}\left(v\frac{\partial T}{\partial x}\right)$$
$$+m\frac{\partial}{\partial y}\left(\frac{\partial a}{\partial T}\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(a\frac{\partial T}{\partial y}\right)=0. \quad (18)$$

Equations (15)–(18) obtained by the variational method are much more complicated than differential equations (1)–(3) of the considered process, but when $m \to 0$ equations (16), (18) and (17) become identical to equations (1), (2) and (3) describing steady-state momentum and thermal laminar boundary layer of incompressible variable properties fluid over a flat plate.

The second of equations (15) gives the boundary condition for Lagrange's multiplier

$$y = y_m \mu_1(x,y) = 0, \quad (19)$$

while the other equations are identically satisfied.

APPROXIMATE SOLUTION OF THE PROBLEM

In order to obtain an approximate solution of the problem under consideration the direct method of variational calculus in the form of partial integration (the method of Kantorovich) will be used. The essence of this method is to presume the shapes of the unknown functions satisfying a reasonable number of boundary conditions. After substituting these functions into action integrals and performing partial integrations, so called reduced action integrals are obtained. The stationarity conditions of reduced action integrals result in Lagrange's equations of the problem.

For the problem described by equations (1), (2) and (3) we shall presume the shapes of the unknown functions $u(x,y)$, $v(x,y)$, $T(x,y)$ and $\mu_1(x,y)$.

Let the solution for the velocity component $u(x,y)$ parallel to the plate be of the form:

$$u(x,y) = U_\infty \phi_1(\lambda), \quad (20)$$

where U_∞ is the free stream velocity and ϕ_1 function of a dimensionless parameter λ given by:

$$\lambda = \frac{y}{f(x)},$$

$f(x)$ being the thickness of the momentum boundary layer.

According to the assumption of the finite thickness of the boundary layer the following boundary conditions can be written:

$$y = y_m = f: \quad u = U_\infty, \quad \frac{\partial u}{\partial y} = 0.$$

The corresponding conditions for the function ϕ_1 are:

$$\lambda = \lambda_m = 1: \quad \phi_1 = 1, \quad \frac{d\phi_1}{d\lambda} = 0.$$

The transverse velocity component $v(x, y)$ is given by the approximate function

$$v = g(x) \cdot N(\lambda) - \varphi(x) R(\lambda), \tag{21}$$

where $N(\lambda)$ and $R(\lambda)$ satisfy the following conditions:

$$\frac{dN}{d\lambda} = N'(\lambda) = \lambda \cdot \phi_1'(\lambda) \quad \text{and} \tag{22}$$

$$\frac{dR}{d\lambda} = R'(\lambda) = \phi_1(\lambda), \tag{23}$$

or in the integral form

$$N(\lambda) = \int \lambda \phi_1'(\lambda) d\lambda + C_1 \tag{24}$$

$$R(\lambda) = \int \phi_1(\lambda) d\lambda + C_2. \tag{25}$$

In order to satisfy the boundary condition $v_{(x,0)} = 0$ according to equation (21) must be:

$$N(0) = 0$$
$$R(0) = 0. \tag{26}$$

The approximate dimensionless temperature profile is of the form:

$$\frac{T - T_w}{T_\infty - T_w} = \phi_2(\eta),$$

where $\eta = [y/\Delta(x)]$, $\Delta(x)$ denoting the thickness of the thermal boundary layer.

The temperature profile $T(x, y)$ can be rewritten in the form:

$$T = T_0 \phi_2(\eta) + T_w, \quad T_0 = T_\infty - T_w. \tag{27}$$

The boundary conditions:

$$y = y_m = \Delta: \quad T = T_\infty, \quad \frac{\partial T}{\partial y} = 0$$

can be written in the form

$$\eta = \eta_m = 1: \quad \phi_2 = 1, \quad \frac{d\phi_2}{d\eta} = 0.$$

We shall assume that kinematic viscosity v and thermal diffusivity a are linear functions of temperature:

$$v = v_\infty(1 + A\theta)$$
$$a = a_\infty(1 + B\theta) \tag{28}$$

where

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}.$$

In the case of heating of an incompressible fluid will be $A < 0$, $B > 0$, while the case $A > 0$, $B < 0$ corresponds to cooling.

Since $\theta = 1 - \phi_2(\eta)$, equation (28) become

$$v = v_\infty[A^* - A\phi_2(\eta)]; \quad A^* = A + 1$$
$$a = a_\infty[B^* - B\phi_2(\eta)]; \quad B^* = B + 1 \tag{29}$$

Lagrange's multiplier can be written in the form

$$\mu_1 = \beta(x) \cdot Q(\lambda), \tag{30}$$

where $Q(\lambda)$ must be chosen to satisfy boundary condition (19).

We shall substitute velocity profiles (20) and (21), temperature profile (27) and Lagrange's multiplier (30) into action integrals (7) and (8), and afterwards we shall perform the integration over thicknesses of momentum and thermal boundary layer. The cases $\Delta < f$ and $\Delta > f$ will be treated separately.

After the integration, the following reduced action integrals are obtained:

$$I_1 = \int_{x_0}^{l} L_1(f, f', \Delta, g, \varphi, \beta, x, m) dx \tag{31}$$

$$I_2 = \int_{x_0}^{l} L_2(f, \Delta, \Delta', g, \varphi, x, m) dx \tag{32}$$

with partial Lagrangians

$$L_1 = \left\{ m \left[\frac{1}{2} \frac{U_\infty^3 f'^2}{f} A_1 - \frac{U_\infty^2 g f'}{f} A_2 + \frac{U_\infty^2 \varphi f'}{f} A_3 \right] - \frac{v_\infty U_\infty^2}{2f} (A^* A_4 - AF) \right\} e^{x/m} - \beta\varphi A_5 + \beta(g - U_\infty f') A_6 \tag{33}$$

$$L_2 = \left\{ m \left[\frac{T_0^2 U_\infty \Delta'^2}{2} F_1 + T_0^2 g \Delta' F_2 + T_0^2 \varphi \Delta' P_1 - \frac{a_\infty B T_0^2 \Delta'}{2\Delta^2} B_1 + \frac{v_\infty U_\infty^2 T_0 \Delta'}{C_p f^2} (A^* F_3 - AF_4) \right] - \frac{a_\infty T_0^2}{2\Delta} (B^* B_2 - BB_3) \right\} e^{x/m}. \tag{34}$$

For $\Delta > f$ reduced action integrals are:

$$J_1 = \int_{x_0}^{l} \mathscr{L}_1(f, f', \Delta, g, \varphi, \beta, x, m) dx \tag{35}$$

$$J_2 = \int_{x_0}^{l} \mathscr{L}_2(f, \Delta, \Delta', g, \varphi, x, m) dx, \tag{36}$$

with partial Lagrangians:

$$\mathscr{L}_1 = \left\{ m \left[\frac{U_\infty^3 f'}{2f} A_1 - \frac{U_\infty^2 g f'}{f} A_2 + \frac{U_\infty^2 \varphi f'}{f} A_3 \right] \right.$$

$$-\frac{\nu_\infty U_\infty^2}{2f}(A^*A_4 - A\psi)\Big\}e^{x/m}$$

$$-\beta\varphi A_5 + \beta(g - U_\infty f')A_6 \quad (37)$$

$$\mathscr{L}_2 = \Big\{m\Big[\frac{T_0^2 U_\infty \Delta'}{2}\psi_1 + T_0^2 g\Delta'\psi_2 + T_0^2\varphi\Delta' P_2$$

$$-\frac{a_\infty B T_0^2 \Delta'}{2\Delta^2}B_1 + \frac{\nu_\infty U_\infty^2 T_0 \Delta'}{C_p f^2}(A^*\psi_3 - A\psi_4)\Big]$$

$$-\frac{a_\infty T_0^2}{2\Delta}(B^*B_2 - BB_3)\Big\}e^{x/m}. \quad (38)$$

In Lagrangians given above constants and functions are defined by the following integrals:

$$A_1 = \int_0^1 \phi_1(\lambda)\phi_1'^2(\lambda)\lambda^2 d\lambda; \quad A_2 = \int_0^1 N(\lambda)\phi_1'^2(\lambda)\lambda\, d\lambda;$$

$$A_3 = \int_0^1 R(\lambda)\phi_1'^2(\lambda)\lambda\, d\lambda; \quad A_4 = \int_0^1 \phi_1'^2(\lambda)d\lambda;$$

$$A_5 = \int_0^1 \phi_1(\lambda)Q(\lambda)d\lambda; \quad A_6 = \int_0^1 \phi_1'(\lambda)Q(\lambda)\lambda\, d\lambda;$$

$$B_1 = \int_0^1 \phi_2'^3(\eta)\eta\, d\eta; \quad B_2 = \int_0^1 \phi_2'^2(\eta)d\eta;$$

$$B_3 = \int_0^1 \phi_2(\eta)\phi_2'^2(\eta)d\eta$$

$$F(f,\Delta) = \int_0^{\Delta/f} \phi_2\left(\frac{f}{\Delta},\lambda\right)\phi_1'^2(\lambda)d\lambda + \int_{\Delta/f}^1 \phi_1'^2(\lambda)d\lambda;$$

$$P_1(f,\Delta) = \frac{1}{\Delta}\int_0^1 R\left(\frac{\Delta}{f},\eta\right)\phi_2'^2(\eta)\eta\, d\eta$$

$$F_1(f,\Delta) = \frac{1}{\Delta}\int_0^1 \phi_1\left(\frac{\Delta}{f},\eta\right)\phi_2'^2(\eta)\eta^2 d\eta;$$

$$F_2(f,\Delta) = -\frac{1}{\Delta}\int_0^1 N\left(\frac{\Delta}{f},\eta\right)\phi_2'^2(\eta)\eta\, d\eta$$

$$F_3(f,\Delta) = \int_0^1 \phi_1'^2\left(\frac{\Delta}{f},\eta\right)\phi_2'(\eta)\eta\, d\eta;$$

$$F_4(f,\Delta) = \int_0^1 \phi_1'^2\left(\frac{\Delta}{f},\eta\right)\phi_2'(\eta)\phi_2(\eta)\eta\, d\eta$$

$$\psi(f,\Delta) = \int_0^1 \phi_2\left(\frac{f}{\Delta},\lambda\right)\phi_1'^2(\lambda)d\lambda;$$

$$\psi_1(f,\Delta) = \frac{1}{\Delta}\left[\int_0^{f/\Delta} \phi_1\left(\frac{\Delta}{f},\eta\right)\phi_2'^2(\eta)\eta^2 d\eta + \int_{f/\Delta}^1 \phi_2'^2(\eta)\eta^2 d\eta\right];$$

$$\psi_2(f,\Delta) = -\frac{1}{\Delta}\int_0^{f/\Delta} N\left(\frac{\Delta}{f},\eta\right)\phi_2'^2(\eta)d\eta,$$

$$P_2 = \frac{1}{\Delta}\int_0^{f/\Delta} R\left(\frac{\Delta}{f},\eta\right)\phi_2'^2(\eta)d\eta;$$

$$\psi_3(f,\Delta) = \int_0^{f/\Delta} \phi_1'^2\left(\frac{\Delta}{f},\eta\right)\phi_2'(\eta)\eta\, d\eta;$$

$$\psi_4(f,\Delta) = \int_0^{f/\Delta} \phi_1'^2\left(\frac{\Delta}{f},\eta\right)\phi_2(\eta)\phi_2'(\eta)d\eta. \quad (39)$$

In equations (31)–(38) the following notation was used

$$f' = \frac{df}{dx}; \quad \Delta' = \frac{d\Delta}{dx}; \quad \phi_1' = \frac{d\phi_1}{d\lambda}; \quad \phi_2' = \frac{d\phi_2}{d\lambda}.$$

Let us consider the case $\Delta < f$ (thermal boundary layer is thinner than the momentum one). Functions f, g, φ, β and Δ are generalized coordinates.

According to the theory of chain-systems and with regard to the connection between functions f, g, φ, β and Δ on one side and u, v, T and μ_1 on the other side, we can consider that generalized coordinates f, g, φ and β belong to the first group and Δ to the second group of coordinates with corresponding Lagrangians L_1 and L_2 respectively. Accordingly when varying reduced action integral (31), functions f, g, φ and β are varied, while the variation of reduced action integral (32) relates to function Δ.

If velocity and temperature, i.e. functions f, g, φ, β and Δ are defined at all boundaries except at $x = l$ and if at $x = l$, for arbitrary values of variations δf and $\delta\Delta$, the following natural conditions:

$$\frac{\partial L_1}{\partial f'}\delta f\Big|_{x=l} = 0; \quad \frac{\partial L_2}{\partial \Delta'}\delta\Delta\Big|_{x=l} = 0 \quad (40)$$

are satisfied, then the conditions of stationarity of reduced action integrals

$$\delta I_1 = 0 \quad \text{and} \quad \delta I_2 = 0$$

result in Euler–Lagrange's equations:

$$\frac{\partial L_1}{\partial f} - \frac{d}{dx}\frac{\partial L_1}{\partial f'} = 0 \quad (41)$$

$$\frac{\partial L_1}{\partial g} - \frac{d}{dx}\frac{\partial L_1}{\partial g'} = 0 \quad (42)$$

$$\frac{\partial L_1}{\partial \varphi} - \frac{d}{dx}\frac{\partial L_1}{\partial \varphi'} = 0 \quad (43)$$

$$\frac{\partial L_1}{\partial \beta} - \frac{d}{dx}\frac{\partial L_1}{\partial \beta'} = 0 \quad (44)$$

$$\frac{\partial L_2}{\partial \Delta} - \frac{d}{dx}\frac{\partial L_2}{\partial \Delta'} = 0. \quad (45)$$

Introducing partial Lagrangians (33) and (34) into equations (40)–(45), dividing them by $e^{x/m}$ and accomplishing the limiting transition $m \to 0$, equations (40), (42) and (43) are identically satisfied, while equations (41) and (45) result in:

$$-\frac{\partial}{\partial f}\left[\frac{\nu_\infty U_\infty^2}{2f}(A^*A_4 - AF)\right] - \frac{U_\infty^3 f'}{f}A_1 + \frac{U_\infty^2 g}{f}A_2 - \frac{U_\infty^2\varphi}{f}A_3 = 0 \quad (46)$$

$$\frac{a_\infty T_0^2}{2\Delta^2}(B^*B_2 - BB_3) - U_\infty T_0^2\Delta' F_1$$

$$- T_0^2 g F_2 - T_0^2 \varphi P_1$$

$$+ \frac{a_\infty T_0^2 B}{2\Delta^2} B_1 - \frac{v_\infty U_\infty^2 T_0}{C_p f^2}(A^*F_3 - AF_4) = 0. \quad (47)$$

Equation (44) gives

$$-\varphi A_5 + (g - U_\infty f')A_6 = 0. \quad (48)$$

This equation is satisfied only when

$$\varphi = 0; \quad g = U_\infty f'. \quad (49)$$

Rearranging equations (46) and (47), using condition (49) we obtain:

$$2U_\infty(A_1 - A_2)ff' + v_\infty\left(AF - Af\frac{\partial F}{\partial f} - A^*A_4\right) = 0$$

$$2U_\infty\Delta^2\Delta' F_1 + 2U_\infty\Delta^2 f' F_2$$

$$+ 2\frac{v_\infty U_\infty^2}{C_p T_0}\frac{\Delta^2}{f^2}(A^*F_3 - AF_4)$$

$$+ a_\infty(BB_3 - BB_1 - B^*B_2) = 0. \quad (50)$$

Applying the conditions of stationarity of reduced action integrals (35) and (36), with partial Lagrangians (37) and (38), the differential equations for the case $\Delta > f$ are obtained:

$$2U_\infty(A_1 - A_2)ff' + v_\infty\left(A\psi - Af\frac{\partial\psi}{\partial f} - A^*A_4\right) = 0$$

$$2U_\infty\Delta^2\Delta'\psi_1 + 2U_\infty\Delta^2 f'\psi_2$$

$$+ 2\frac{v_\infty U_\infty^2}{C_p T_0}\frac{\Delta^2}{f^2}(A^*\psi_3 - A\psi_4)$$

$$+ a_\infty(BB_3 - BB_1 - B^*B_2) = 0. \quad (51)$$

We shall assume velocity and temperature profiles of Targ [5] in the form:

$$\phi_1(\lambda) = \tfrac{3}{2}\lambda - \tfrac{1}{2}\lambda^3$$
$$\phi_2(\eta) = \tfrac{3}{2}\eta - \tfrac{1}{2}\eta^3, \quad (52)$$

which satisfy the following boundary conditions:

$y = 0,\ \lambda = 0,\ \eta = 0$:

$$\phi_1 = 0,\quad \phi_2 = 0,\quad u = 0,\quad T = T_w$$

$$y = f:\ \lambda = 1:\ \phi_1 = 1,\quad \phi_1' = 0,\quad u = U_\infty,\quad \frac{\partial u}{\partial y} = 0$$

$$y = \Delta,\ \eta = 1: \phi_2 = 1,\quad \phi_2' = 0,\quad T = T_\infty,\quad \frac{\partial T}{\partial y} = 0$$

From equations (24) and (26) we obtain

$$N(\lambda) = \tfrac{3}{8}(2\lambda^2 - \lambda^4).$$

Substituting functions ϕ_1, ϕ_2 and N into (39) we get the following constants and functions:

$$A_1 = \frac{39}{320};\quad A_2 = \frac{9}{160};\quad A_4 = \frac{6}{5}$$

$$B_1 = \frac{27}{64};\quad B_2 = \frac{6}{5};\quad B_3 = \frac{33}{64}$$

$$F = \frac{9}{8}\left(-\frac{3}{4}\frac{\Delta}{f} + \frac{1}{6}\frac{\Delta^3}{f^3} - \frac{1}{40}\frac{\Delta^5}{f^5}\right) + \frac{5}{6};$$

$$F_1 = \frac{9}{32}\left(\frac{1}{2}\frac{1}{f} - \frac{1}{15}\frac{\Delta^2}{f^3}\right);$$

$$F_2 = \frac{9}{32}\left(-\frac{1}{4}\frac{\Delta}{f^2} + \frac{1}{20}\frac{\Delta^3}{f^4}\right);$$

$$F_3 = \frac{27}{8}\left(\frac{1}{4} - \frac{1}{6}\frac{\Delta^2}{f^2} + \frac{1}{24}\frac{\Delta^4}{f^4}\right);$$

$$F_4 = \frac{27}{6}\left(\frac{12}{35} - \frac{88}{315}\frac{\Delta^2}{f^2} + \frac{52}{693}\frac{\Delta^4}{f^4}\right)$$

$$\psi = \frac{9}{16}\left(\frac{f}{\Delta} - \frac{1}{12}\frac{f^3}{\Delta^3}\right);$$

$$\psi_1 = \frac{9}{8\Delta}\left(\frac{16}{105} - \frac{1}{12}\frac{f^3}{\Delta^3} + \frac{1}{20}\frac{f^5}{\Delta^5} - \frac{3}{280}\frac{f^7}{\Delta^7}\right);$$

$$\psi_2 = \frac{27}{32\Delta}\left(-\frac{1}{3}\frac{f^2}{\Delta^2} + \frac{5}{12}\frac{f^4}{\Delta^4} - \frac{3}{20}\frac{f^6}{\Delta^6}\right);$$

$$\psi_3 = \frac{27}{48}\left(\frac{f^2}{\Delta^2} - \frac{1}{4}\frac{f^4}{\Delta^4}\right);$$

$$\psi_4 = \frac{27}{16}\left(\frac{8}{35}\frac{f^3}{\Delta^3} - \frac{32}{315}\frac{f^5}{\Delta^5} + \frac{8}{693}\frac{f^7}{\Delta^7}\right). \quad (53)$$

Substituting necessary functions and constants (53) into equations (50) and (51) we obtain:

$$\frac{21}{160}U_\infty ff' + v_\infty$$
$$\times\left[\frac{9}{8}A\left(-\frac{3}{2}\frac{\Delta}{f} + \frac{2}{3}\frac{\Delta^3}{f^3} - \frac{3}{20}\frac{\Delta^5}{f^5}\right) - \frac{6}{5}\right] = 0$$

$$\frac{9}{16}U_\infty\Delta\Delta'\left(\frac{1}{2}\frac{\Delta}{f} - \frac{1}{15}\frac{\Delta^3}{f^3}\right)$$
$$+ \frac{3}{16}U_\infty\Delta f'\left(-\frac{1}{4}\frac{\Delta^2}{f^2} + \frac{1}{20}\frac{\Delta^4}{f^4}\right)$$
$$+ \frac{27}{8}\frac{v_\infty U_\infty^2}{C_p T_0}\left[\left(\frac{11}{70}A + \frac{1}{2}\right)\frac{\Delta^2}{f^2} - \left(\frac{17}{315}A + \frac{1}{3}\right)\frac{\Delta^4}{f^4}\right.$$
$$\left. + \left(\frac{23}{2772}A + \frac{1}{12}\right)\frac{\Delta^6}{f^6}\right] - a_\infty\left(\frac{177}{160}B + \frac{6}{5}\right) = 0. \quad (54)$$

$$\frac{21}{160}U_\infty ff' + v_\infty\left[\frac{3}{32}A\frac{f^3}{\Delta^3} - (A+1)\frac{6}{5}\right] = 0$$

$$\frac{9}{4}U_\infty\Delta\Delta'\left(\frac{16}{105} - \frac{1}{12}\frac{f^3}{\Delta^3} + \frac{1}{20}\frac{f^5}{\Delta^5} - \frac{3}{280}\frac{f^7}{\Delta^7}\right)$$

$$+\frac{27}{16}U_\infty\Delta f'\left(-\frac{1}{3}\frac{f^2}{\Delta^2}+\frac{5}{12}\frac{f^4}{\Delta^4}-\frac{3}{20}\frac{f^6}{\Delta^6}\right)$$

$$+\frac{9}{8}\frac{\nu_\infty U_\infty^2}{T_0C_p}\left[(A+1)-\frac{24}{35}A\frac{f}{\Delta}-\frac{1}{4}(A+1)\frac{f^2}{\Delta^2}\right.$$

$$\left.+\frac{32}{105}A\frac{f^3}{\Delta^3}-\frac{8}{231}A\frac{f^5}{\Delta^5}\right]$$

$$+a_\infty\left[\frac{3}{32}B-(B+1)\frac{6}{5}\right]=0. \qquad (55)$$

It is convenient to introduce new dimensionless variables

$$\xi=\frac{x}{l};\quad \varphi=\frac{f^2}{l^2};\quad Y=\frac{\Delta}{f}, \qquad (56)$$

where l indicates the characteristic length of the plate. According to (56) will be:

$$ff'=\frac{1}{2}l\frac{\mathrm{d}\varphi}{\mathrm{d}\xi};\quad \Delta\Delta'=\frac{1}{2}l\left(Y^2\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}+2\varphi Y\frac{\mathrm{d}Y}{\mathrm{d}\xi}\right);$$

$$\Delta f'=\frac{1}{2}l\,Y\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}. \qquad (57)$$

Rearranging equations (54) and (55) using equations (56) and (57) and introducing Reynolds [$Re_\infty = (U_\infty l/\nu_\infty)$], Prandtl [$Pr_\infty = (\nu_\infty/a_\infty)$] and Eckert number {$Ec=[U_\infty^2/C_p(T_w-T_\infty)]$}, we obtain the differential equations of the problem:

for $\Delta\le f$

$$\frac{21}{320}Re_\infty\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}+\frac{9}{8}A\left(-\frac{3}{2}Y+\frac{2}{3}Y^3-\frac{3}{20}Y^5\right)-\frac{6}{5}=0$$

$$\left(\frac{1}{4}Y^3-\frac{1}{60}Y^5\right)\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}+\varphi\left(Y^2-\frac{2}{15}Y^4\right)\frac{\mathrm{d}Y}{\mathrm{d}\xi}$$

$$-12\frac{Ec}{Re_\infty}\left[\left(\frac{11}{7}A+\frac{1}{2}\right)Y^2\right.$$

$$\left.-\left(\frac{17}{315}A+\frac{1}{3}\right)Y^4+\left(\frac{23}{2772}A+\frac{1}{12}\right)Y^6\right]$$

$$-\frac{32}{9}\frac{1}{Re_\infty Pr_\infty}\left(\frac{177}{160}B+\frac{6}{5}\right)=0. \qquad (58)$$

For $\Delta\ge f$:

$$\frac{21Re_\infty}{320}\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}+\frac{3}{32}AY^{-3}-(A+1)\frac{6}{5}=0$$

$$\left(\frac{64}{315}Y^2-\frac{4}{9}Y^{-1}+\frac{29}{60}Y^{-3}-\frac{23}{140}Y^{-5}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}$$

$$+\frac{8}{3}\varphi\left(\frac{16}{105}Y-\frac{1}{12}Y^{-2}-\frac{3}{280}Y^{-6}\right)\frac{\mathrm{d}Y}{\mathrm{d}\xi}$$

$$-\frac{4}{3}\frac{Ec}{Re_\infty}\left[(A+1)-\frac{24}{35}AY^{-1}-\frac{1}{4}(A+1)Y^{-2}\right.$$

$$\left.+\frac{32}{105}AY^{-3}-\frac{8}{231}AY^{-5}\right]$$

$$-\frac{32}{27Re_\infty Pr_\infty}\left(\frac{177}{160}B+\frac{6}{5}\right)=0. \qquad (59)$$

The solution of the system of equations (58) and (59) is possible only by numerical methods, however there are some difficulties relating to the boundary condition at the edge of the plate. If the plate is heated along its entire length, rather than at $x=0, f=0$ and $\Delta=0$, thus causing $Y=\Delta/f=0/0$, $x=0$ becomes singularity point. In order to avoid this difficulty we shall assume that there is an unheated initial length x_0 of the plate, so that thermal boundary layer starts at $\xi_0 = (x_0/l)$ (Fig. 1). In this paper the value $\xi_0 = 0.1$ is taken in order to compare the results of these calculations with results in [6]. Differential equations (58) describe the process in the interval $0\le x\le x_1$ (Fig. 1) (or in dimensionless form $0\le\xi\le\xi_1$, where $\xi_1 = x_1/l$), i.e. in the region where $\Delta(x)\le f(x)$. The boundary conditions in this region are:

$$\xi=0;\quad \varphi=0$$

$$0\le\xi\le 0.1:\quad Y\equiv 0. \qquad (60)$$

The second of the boundary conditions (60) makes it possible to solve independently the first equation in (58). In the interval $0.1<\xi\le\xi_1$, equations (58) are coupled.

After solving the system (58) the coordinates of the point M_1 (Fig. 1), for which the values of dimensionless variables are $\xi=\xi_1$, $\varphi=\varphi(\xi_1)$ and $Y=1$, can be determined. These are the boundary conditions for solving system of equation (59) for $\Delta\ge f$.

For the numerical solution of the system of equations (58) and (59) using the method of England [7] the following fluid properties and flow conditions were supposed:

$Re_\infty = 80000$, $Pr_\infty = 2$, $A=-0.5$, $B=0.1$.

Since the object of this paper was to establish the influence of viscous dissipation on the formation of momentum and thermal boundary layers over a flat

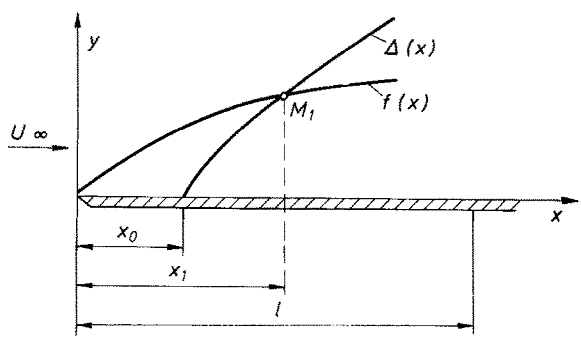

FIG. 1. Momentum and thermal boundary layers over a flat plate.

plate, the Eckert number was varied from 0 to 2.0 ($Ec = 0; 0.2; 0.45; 0.5; 1.0; 2.0$).

Numerical solutions give the dimensionless thicknesses of momentum and thermal boundary layer along the plate:

$$f^* = \frac{f}{l} \sim \sqrt{\varphi}$$

$$\Delta^* = \frac{\Delta}{l} = Y\sqrt{\varphi}.$$

The local Nusselt number is defined by the expression:

$$Nu_x = \frac{x}{T_\infty - T_w}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{x}{T_0}\left(\frac{\partial T}{\partial y}\right)_{y=0}.$$

According to equations (27) and (52) the local Nusselt number can be calculated from the expression

$$Nu = \frac{3}{2}\frac{x}{\Delta} = \frac{3}{2}\frac{\xi}{\Delta^*},$$

or in terms of ξ, Y and φ

$$Nu = \frac{3}{2}\frac{\xi}{Y\sqrt{\varphi}}.$$

The local coefficient of skin friction is given by:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{\mu_w\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\frac{1}{2}\rho U_\infty^2}.$$

Since

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{U_\infty \phi_1'}{f}\right)_{\lambda=0} = \frac{3U_\infty}{2f}$$

and

$$\frac{\mu_w}{\rho} = (v)_{y=0} = v_\infty(1 + A\theta)_{\theta=1} = v_\infty(1 + A),$$

the final expression for c_f is as follows

$$c_f = \frac{3(1 + A)}{Re_\infty f^*} = \frac{3(1 + A)}{Re_\infty \sqrt{\varphi}}.$$

Plots of functions f^*, Δ^*, Nu_x and c_f for different values of Eckert number Ec are given in Figs. 2–7. Figure 8 shows the influence of Eckert number on local Nusselt number along the plate.

DISCUSSIONS AND CONCLUSIONS

According to Fig. 2, the results obtained for the case without viscous dissipation ($Ec = 0$) are in excellent agreement with those from [6] and that is a confirmation of the reliability of the variational method applied.

From Figs. 2–7 it is evident that viscous dissipation has almost no influence to the thickness of momentum boundary layer and consequently to the skin friction coefficient.

According to Figs. 2–7 the thickness of thermal

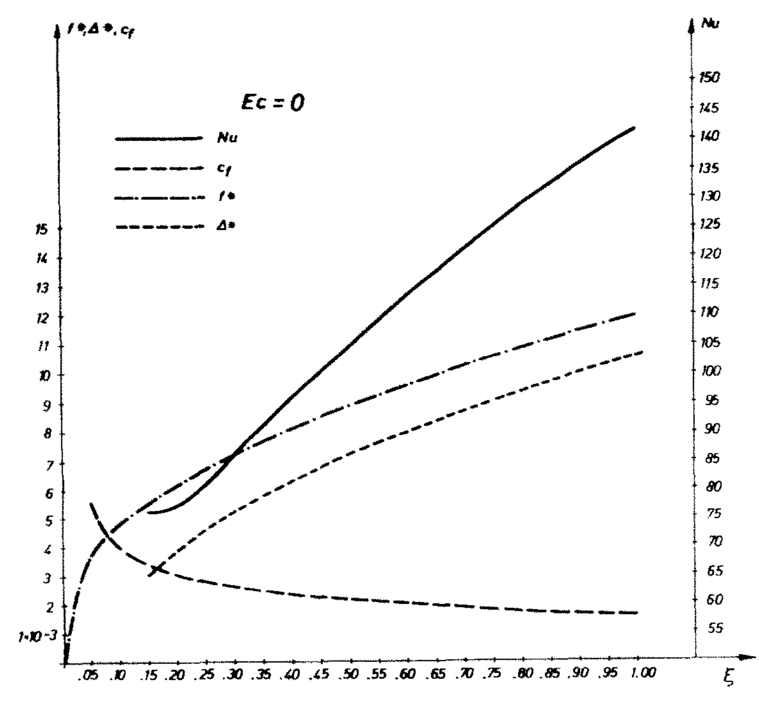

FIG. 2. Plots of functions Nu, c_f, f^* and Δ^* for the case $Ec = 0$.

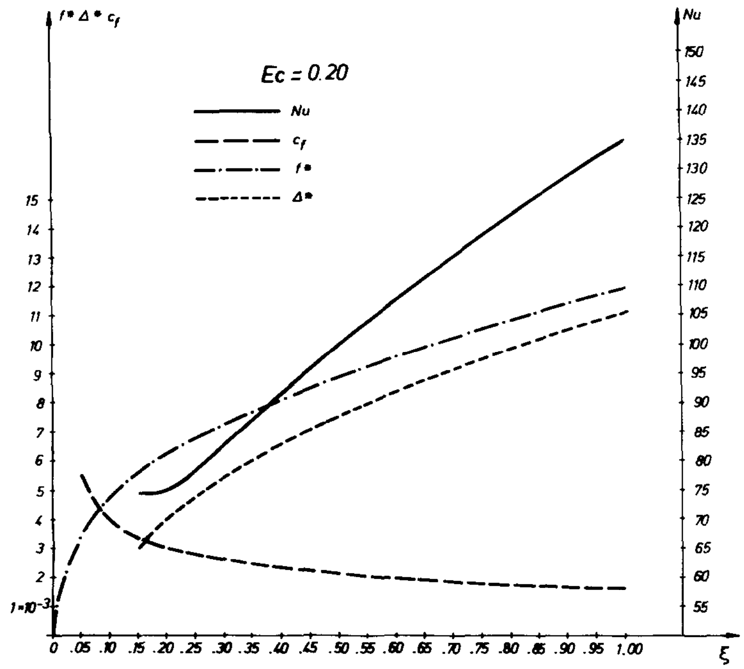

FIG. 3. Plots of functions Nu, c_f, f^* and Δ^* for the case $Ec = 0.20$.

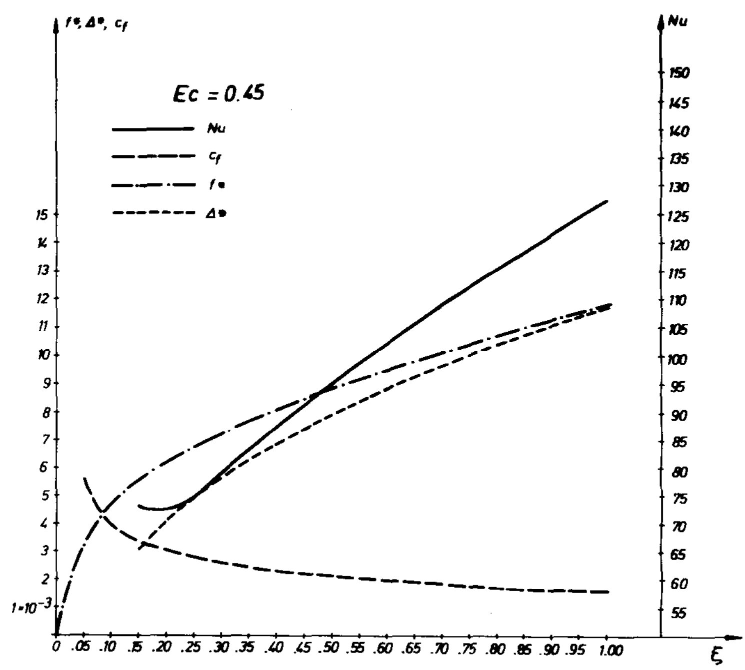

FIG. 4. Plots of functions Nu, c_f, f^* and Δ^* for the case $Ec = 0.45$.

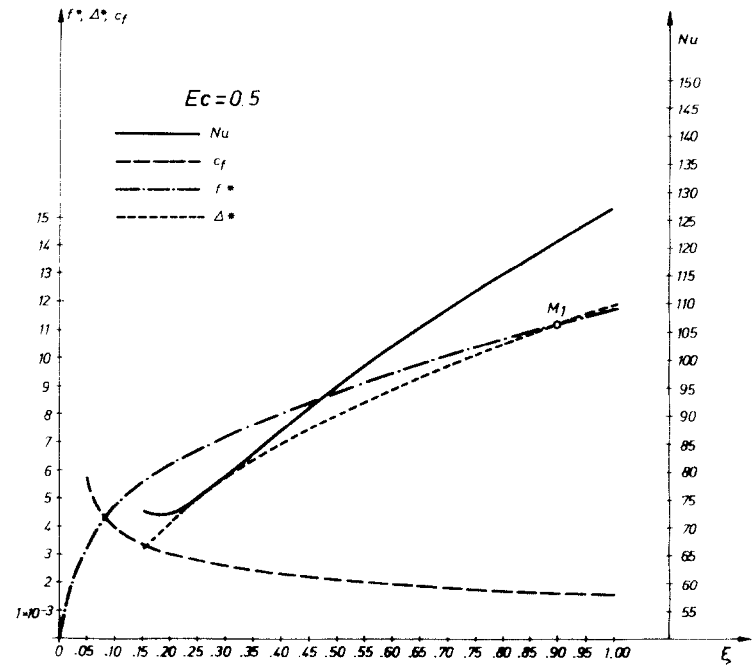

FIG. 5. Plots of functions Nu, c_f, f^* and Δ^* for the case $Ec = 0.50$.

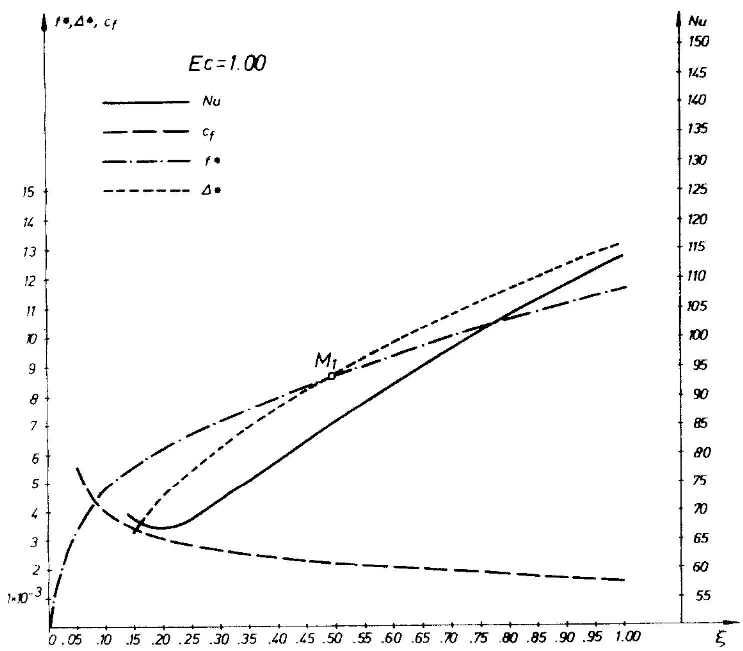

FIG. 6. Plots of functions Nu, c_f, f^* and Δ^* for the case $Ec = 1.00$.

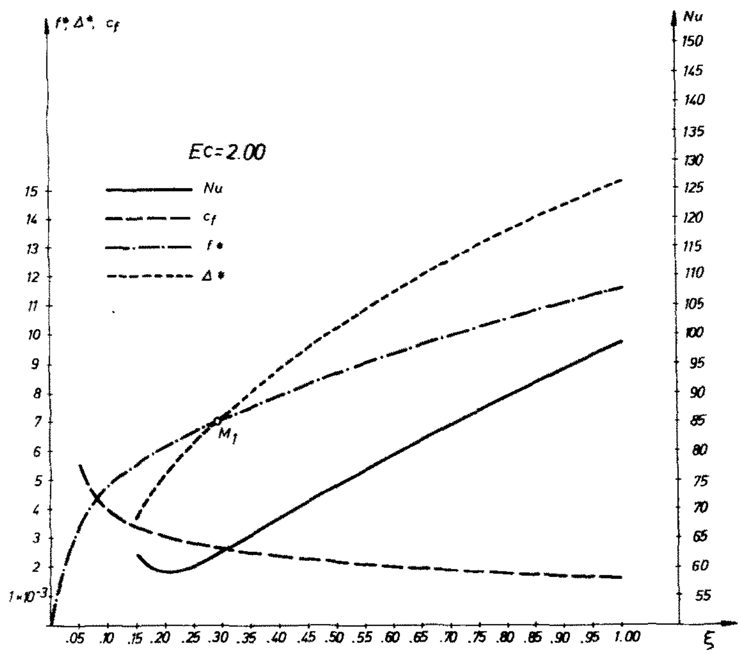

FIG. 7. Plots of functions Nu, c_f, f^* and Δ^* for the case $Ec = 2.00$.

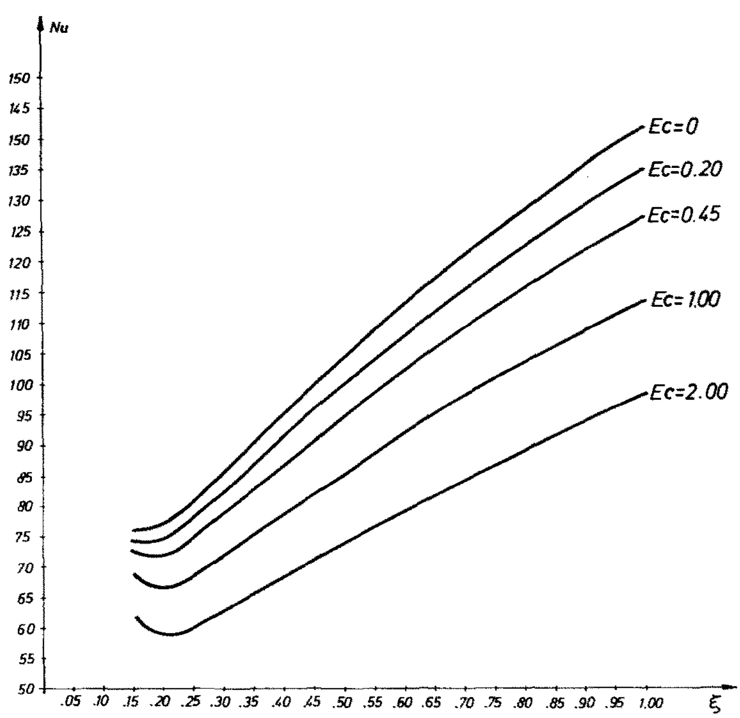

FIG. 8. Influence of Eckert number on local Nusselt number along the plate.

boundary layer increases with increasing Eckert number. This is due to the decrease of the temperature gradient because of the frictional heat generation. The increase of the thermal boundary layer thickness causes the decrease of the local Nusselt number as can be seen from Fig. 8. Figure 8 also shows that with increasing Eckert number there is a more emphasised minimum of the Nusselt number. This is due to the faster increase of thermal boundary layer thickness at the inlet region of the plate with increasing Ec.

A very important conclusion which can be drawn from the results of this work is related to the fact that Prandtl number $Pr > 1$ does not always mean that the thermal boundary layer is thinner than the momentum one. As it can be seen from Figs. 2–7 the assumption $\Delta < f$ is correct as long as the Eckert number is lower than Ec_0 (in this work $Ec_0 \approx 0.45$). When $Ec > Ec_0$ the thermal boundary layer can be thicker than the momentum boundary layer even though $Pr > 1$.

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UNE APPROCHE VARIATIONNELLE DU PROBLEME DE LA CONVECTION LAMINAIRE STATIONNAIRE AVEC DISSIPATION DANS UN FLUIDE A PROPRIETES VARIABLES

Résumé — On utilise un principe variationnel de Hamilton pour résoudre le problème de la convection stationnaire avec dissipation dans un fluide à propriétés variables s'écoulant sur une plaque plane avec une longueur préalable non chauffée. La diffusivité thermique et la viscosité sont supposés fonctions linéaires de la température. Deux équations différentielles sont obtenues et résolues numériquement pour des nombres d'Eckert allant de 0 à 2 en utilisant un ordinateur. Les résultats montrent que la valeur du nombre d'Eckert a une influence considérable sur le transfert thermique, tandis que le changement sur le coefficient de frottement est négligeable. Pour des valeurs élevées du nombre d'Eckert, à une certaine distance du bord de la plaque, la couche limite thermique devient plus épaisse que la couche de quantité de mouvement même pour un fluide à nombre de Prandtl supérieur à l'unité.

VARIATIONSRECHNERISCHE BEHANDLUNG DES PROBLEMS DER STATIONÄREN LAMINAREN KONVEKTION MIT DISSIPATION IN EINEM FLUID MIT VERÄNDERLICHEN STOFFWERTEN

Zusammenfassung—Ein Variationsprinzip nach Hamilton wurde auf das Problem der stationären Konvektion mit Dissipation in einem Fluid mit veränderlichen Stoffwerten angewendet, das über eine ebene Platte mit einem unbeheizten Anfangsabschnitt strömt. Temperaturleitzahl und Viskosität wurden als lineare Funktionen der Temperatur angenommen. Aus der Variationsrechnung wurden zwei gewöhnliche Differentialgleichungen hergeleitet und dann für Eckert-Zahlen im Bereich von 0 bis 2 mit einem Digitalrechner numerisch gelöst. Die Ergebnisse zeigen, daß der Wert der Eckert-Zahl einen beachtlichen Einfluß auf der Wärmeübergang hat, während die Änderung des Reibungsbeiwertes vernachlässigbar ist. Bei höheren Werten der Eckert-Zahl wird ab einem gewissen Abstand von der Plattenanströmkante die thermische Grenzschicht dicker als die Impulsgrenzschicht, selbst für ein Fluid mit einer Prandtl-Zahl größer als eins.

ВАРИАЦИОННЫЙ ПОДХОД К ПРОБЛЕМЕ СТАЦИОНАРНОЙ ЛАМИНАРНОЙ КОНВЕКЦИИ ПРИ НАЛИЧИИ ДИССИПАЦИИ В ЖИДКОСТИ С ПЕРЕМЕННЫМИ СВОЙСТВАМИ

Аннотация — Вариационный принцип Гамильтона используется для решения задачи стационарной конвекции при наличии диссипации в потоке жидкости с переменными свойствами над плоской пластиной с необогреваемым начальным участком. Предполагается, что температуропроводность и вязкость являются линейными функциями температуры. Из вариационного принципа получены два обыкновенных дифференциальных уравнения и на вычислительной машине получено их численное решение для значений числа Эккерта от 0 до 2. Результаты показывают, что величина числа Эккерта оказывает большое влияние на перенос тепла и почти не влияет на коэффициент трения. При более высоких значениях числа Эккерта высота теплового пограничного слоя на определенном расстоянии от края пластины начинает превышать толщину динамического пограничного слоя даже у жидкостей с числом Прандтля больше единицы.